Online algorithms for dummies

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Journées CALAMAR

(Journées Combinatoires des Alpes, des Littoraux Atlantique et Méditéranéen, d'Auvergne et du Rhône)

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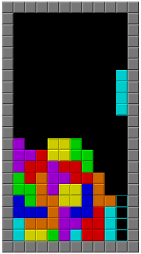
Today : online algorithms

- Talk 1 : Introduction to online algorithms - Nicolas Bousquet (LIRIS).
- Talk 2 : Online algorithms with predictions - Bertrand Simon (IN2P3).
- Talk 3 : Online edge coloring - Clément Legrand-Duchesne (LaBRI).

What is an online algorithm ?

- Input arrives sequentially over time (arrival order).
- Decisions must be taken without the knowledge of the future input.
- Decisions are irrevocable.



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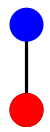

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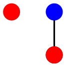

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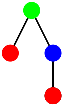

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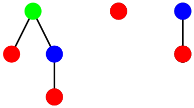

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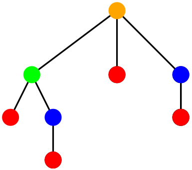

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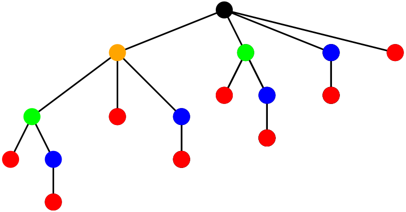

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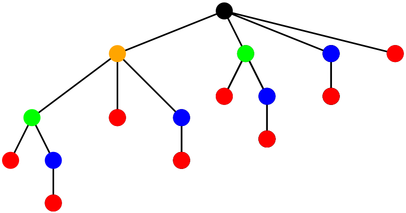

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Typical question : Can we find an algorithm that "approximates" the quality of the best offline algorithm ?

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Adaptive adversary. Knows the algorithm and all the choices performed so far and chooses the **next action**. (stronger adversary)

Two levels of such adversaries

Deterministic vs randomized

Two types of online algorithms : deterministic or randomized !

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Remark :
Oblivious and adaptive adversaries are equivalent for deterministic algorithms.

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Performace of an online algorithm Given a maximization problem, I an instance, an algorithm is :

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Remark :

- $\alpha \leq 1$ and if $\alpha = 1$ we have an almost optimal algorithm.
- For a minimization function we can twist the definition
- For a deterministic algorithm, we are just looking for the worst instance. For randomized algorithms, we look for the worst possible expected size.

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Competitive ratio : $\frac{n}{B} \rightarrow +\infty$ when n tends to infinity.

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Proof : Let k the integer where the opponent decide to stop.

- If $k \leq B-1$, the optimal strategy consists in renting and that's what we do.
- If $k \geq B$, the optimal strategy (of cost B) consists in buying skis at day 1. The break-even strategy has cost $2B-1$.

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Proof :

- Determinist strategy : choose an integer t.
- Opponent strategy : either choose $t' < t$ or $t' = t$.
- Make calculations...

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There exists a $(1-\frac{1}{e})^{-1}$-competitive randomized online algorithm for ski rental.

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A deterministic strategy $\mathcal{S}_1$ is dominated by $\mathcal{S}_2$ if for **every** possible choice of t by the adversary, the cost($\mathcal{S}_1$)$\geq$cost($\mathcal{S}_2$).

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No dominated strategy has a positive probability in an optimal mixed strategy.

Game theory perspective

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Cost of the strategies depending on the ending time

Str. / Stop	1	2	3	4
$\mathcal{A}_1$	B	B	B	B
$\mathcal{A}_2$	1	B+1	B+1	B+1
$\mathcal{A}_3$	1	2	B+2	B+2
$\mathcal{A}_4$	1	2	3	B+3

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Similarly, if he decides to stop at step 2. The optimal cost is 2 and the expected cost of the strategy is $Bp_1 + (B+1)p_2 + 2p_3 + 2p_4$.

LP formulation

$$
\begin{aligned}
\min x & & \\
Bp_1 + p_2 + p_3 + p_4 & \leq & x \\
\frac{1}{2}(Bp_1 + (B+1)p_2 + 2p_3 + 2p_4) & \leq & x \\
\frac{1}{3}(Bp_1 + (B+1)p_2 + (B+2)p_3 + 3p_4) & \leq & x \\
\frac{1}{4}(Bp_1 + (B+1)p_2 + (B+2)p_3 + (B+3)p_4) & \leq & x \\
p_1 + p_2 + p_3 + p_4 & = & 1
\end{aligned}
$$

Best solution : $1/(1-\frac{1}{4})^4 \rightarrow (1-\frac{1}{e})^{-1}$.

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Yao's Lemma

Assume that there is a distribution $\mathcal{D}$ over instances of Π such that every deterministic online algorithm has expected competitive ratio at least μ. Then, the competitive ratio of every randomized online algorithm for Π is at least μ.

What about adaptive adversaries ?

You'll continue skiing until you decide to buy your skis !

$\rightarrow$ We cannot improve the 2-competitive factor.

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Model : Vertices arrive one by one (with their edges to already appeared vertices).

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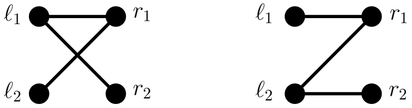

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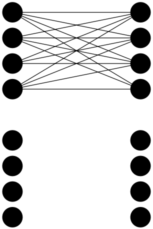

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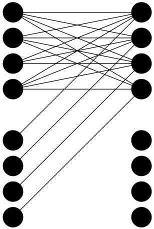

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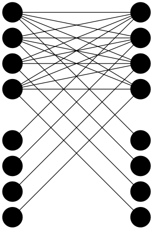

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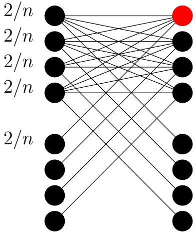

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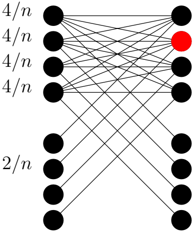

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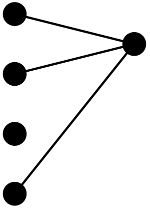

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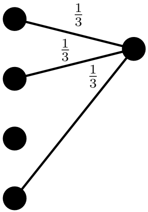

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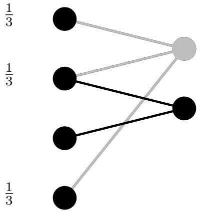

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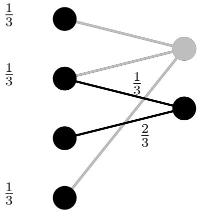

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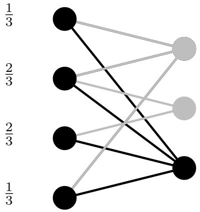

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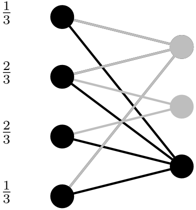

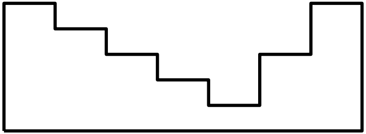

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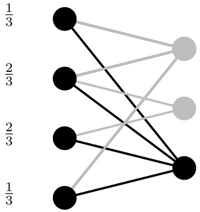

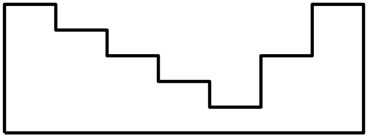

Mathematically :

- $d(i) = \sum_{(i,j)\in E} x_{ij}$. (Initial level of water on ℓ_i)
- Find $\ell_j = \min_{i\in N(j)} d(i) + r_i$ such that $\sum r_i = 1$ (with $\ell_j \leq 1$).
  (Final level of water)
- Update x_{ij} : increase it by $\ell_j - d(i) = r_i$ (or 0 if neg.).

Primal-dual analysis

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Fractional matching :

$$\max \sum_{(i,j)\in E} x_{ij}$$

$$\text{soumis à}$$

$$\sum_{j/(i,j)\in E} x_{ij} \leq 1 \qquad \forall i \in L$$

$$\sum_{i/(i,j)\in E} x_{ij} \leq 1 \qquad \forall j \in R$$

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- Start with a solution where $x_{ij} = 0$ (with no constraint since $G = \emptyset$).
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Each time a vertex is added, we update :

$$\begin{cases} \alpha_i = g(d(i)) \\ \beta_j = 1 - g(\ell(j)) \end{cases}$$

where $g(y) = \frac{e^y - 1}{e - 1}$

Analysis (cont.)

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By Weak Duality theorem, it provides a $\frac{e}{e-1}$-approximation algorithm.

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How can we prove such a thing ?

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What increases in the primal :

$$C = \sum_{i \in N(j)} r_i = \sum_{i \in N(j)} \ell_j - d(i)$$

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$\Rightarrow g$ is the function satisfying $1 - g + g' = \frac{e}{e-1}$.

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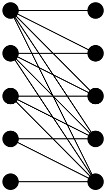

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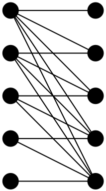
No deterministic algorithm can behave well against all the permutations of the RHS of the half graph.



Randomized Online Bipartite Matching

Model : Vertices of L are there from the beginning (offline vertices).
Vertices of R arrive one after another (online vertices).

Reminder : No deterministic algorithm can beats competitive ratio $\frac{1}{2}$.

Randomized algorithm

Theorem (Karp, Vazirani, Vazirani '90, Goel, Mehta'08)

There exists a $(1 - \frac{1}{e})$-competitive randomized algorithm for online bipartite matching.

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Algorithm RANKING

Choose a random ordering σ of A.

When a vertex of B arrives, match it with its largest (in σ) available neighbor in A.

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Theorem (Karp, Vazirani, Vazirani '90, Goel, Mehta'08)

There exists a $(1 - \frac{1}{e})$-competitive randomized algorithm for on-line bipartite matching.

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Two proofs :

- Primal dual approach (Devanur, Jain, Kleinberg '13)
- With a "typical" probabilistic argument KVV'90, GM'08

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For the analysis : instead of a ranking, we associate to each vertex i of L a random real Y_i in $[0,1]$.

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(Very sketchy) flavour of the proof :

- We will define some (randomized) α_i, β_j when (i, j) are matched.
- $\alpha_i = \frac{e}{e-1} h(Y_i)$ and $\beta_i = \frac{e}{e-1}(1 - h(Y_i))$.
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Follows from properties of $h(y) = e^{y-1}$ close to the ones of the previous proof.

Thanks for your attention !